Ground State for the Quark Mass Hierarchy and Mixings

I.T.Dyatlov
Petersburg Nuclear Physics Institute
Gatchina, St.Petersburg 188350, Russia
e-mail: dyatlov@lnpi.spb.su

February 7, 2008

In the last few years the Nambu-Jona-Lasinio (NJL) model [1] for dynamical chiral violation was widely used to study the possible influence of high energy phenomena on the formation of the standard model. At small distances ($M^{-1} \ll M_w^{-1}$), strong NJL forces were assumed to couple only with the heaviest quarks [2], in which case electroweak symmetry breaking and fermion mass generation were involved via a top-quark condensate. The method [3] suggested a new way to relate high and low energies and obtain testable predictions.

However, few attempts are reported to have been made to involve low masses and quark mixing [4]. The initial separation of the top sector only complicates the understanding.

In the present paper we will extend and develop the small distance part of the scheme suggested in [2, 3]. For the purpose we will extend the NJL equation into the approximation next to the leading N_c one ($N_c \gg 1$ is the number of colors). At first we will show that under certain conditions an NJL system symmetric in n flavors may spontaneously choose the heaviest state out of n initially equal massless quarks. The resulting ground state of one massive m_1 and $(n-1)$ massless quarks is the very one to develop the mass hierarchy and mixings. Secondly, we will determine conditions in which hierarchy and mixing can appear. Here the both are considered as small distance phenomena. Therefore they are to be reflected in the gap

equation for induced masses by flavor dependent NJL coupling constants. The ground state being asymmetric, it is sufficient to have a tiny violation of the original flavor symmetry to have both phenomena generated at the level required. The asymmetric part of the NJL coupling must meet tough restrictions ($\lesssim m_1^2/M^2$ or N_c^{-1}) to yield exactly the mass hierarchy. The same conditions strongly suppress neutral flavor changings.

The delicate problem of the scheme [2, 3] is the phenomenological necessity of fine tuning. It is required that the theory should closely approach the critical point if the heaviest fermion mass $m_1 \ll M$. Now m_1/M will not determine fine-tuning conditions. They can be restricted by the small but constant factor N_c^{-1}.

The interaction we will consider is similar to that in [5, 6]:

$$V = \lambda_{ii'} \left(\bar{\Psi}^c_{Li} \Psi^c_{Ri'} \right) \left(\bar{\Psi}^{c'}_{Ri'} \Psi^{c'}_{Li} \right), \qquad \lambda_{ii'} \sim M^{-2}. \tag{1}$$

Chiral quarks have n flavors (generations) i, i' and N_c colors c, c'. In the general case $\lambda_{ii'} \neq \lambda_{i'i}$. The theory has the symmetry $SU(N_c) \times [U_L(1)]^n \times [U_R(1)]^n$. Therefore mixing between flavors may only be spontaneous. One may believe that Eq.(1) is not an arbitrary chosen example, but that at low energy ($E < M$) it could simulate quite a realistic high energy ($E > M$) situation [5, 7].

We intend to use two successive N_c approximation in the gap equation instead of the leading approximation commonly used up to now. The gap equation for the mass matrix $\Sigma_{ii'}$ is (Fig.1):

$$\begin{aligned}
\Sigma_{ii'} &= \frac{\beta_{ii'}}{M^2} \int \frac{d^4p}{\pi^2 i} \frac{1}{4} Sp\, G_{ii'}(p, \Sigma) f \\
&+ \frac{1}{2N_c} \sum_{i_1 i'_1} \int \frac{d^4q}{\pi^2 i} G_{i_1 i'_1}(-q, \Sigma) B^{i' i'_1}_{i i_1}(q^2) \frac{\beta_{i' i'_1}}{M^2}, \qquad (2) \\
\beta_{ii'} &= \frac{\lambda_{ii'} M^2 N_c}{8\pi^2} = \text{const}, \quad \text{when} \quad N_c \to \infty;
\end{aligned}$$

f is the cut-off function. We shall use the simplest form: $f = \vartheta(M^2 - |p^2|)$. The cut-off form does not influence qualitative results in case we deal with divergences contained in simple loops. The sought-for quark propagator is taken in the form:

$$G^{-1}_{ii'}(p, \Sigma) = \Sigma_{ii'} - \hat{p}\delta_{ii'}. \tag{3}$$

Therefore, in Eq.(2) the external momentum P (Fig.1b) is assumed equal to zero.

In Eq.(2), the quark-antiquark amplitude $N_c^{-1}B\beta/M^2$ can be calculated as a plait of simple loops (Fig.1b). This basic statement follows from the analysis of possible multiloop contributions. When Eq.(2) is valid the general formula for B is

$$B \sim \frac{M^2}{m_1^2}\left[1+\frac{1}{N_c}f_1\left(\frac{m_1^2}{M^2}\right)+\frac{1}{N_c^2}f_2\left(\frac{m_1^2}{M^2}\right)+\cdots\right] \tag{4}$$

at $m_1^2 \sim q^2$ and $m_1^2 \ll M^2$. We do not take into account the powers β^k, as $\beta =$const. The first term represents a sum of simple loops.

Some danger could have appeared if N_c^{-1} was compensated by the large M^2/m_1^2 factor. But such contributions do not exist. Then, the utmost we could obtain is $f_1 \sim f_2 \sim \ldots \sim 0(1)$. However, a part of these contributions becomes important in the N_c^{-1} order, Eq.(2). Their role will be discussed below.

The sum of simple loops can be written in a matrix form:

$$N_c^{-1}B\beta = N_c^{-1}(1-A)^{-1}\beta, \qquad \beta = \begin{pmatrix} 0 & \beta_{ii'} \\ \beta_{ii'} & 0 \end{pmatrix}, \tag{5}$$

where A, B, β are matrices in the chirality and flavor spaces:

$$\begin{aligned} A^{\alpha\beta}_{ii_1,i'i'_1}(q) &= \int \frac{d^4p}{\pi^2 i}\frac{1}{2}Sp\left[\frac{\beta^{\alpha\delta}_{ii_1}}{M^2}O_\delta G_{ii'}(p,\Sigma)O_\beta G_{i_1i'_1}(p-q,\Sigma)\right]f, \\ O_\pm &= \frac{1}{2}(1\pm\gamma_5). \end{aligned} \tag{6}$$

The coupling constants we are interested in are almost independent of flavor indices (see Eq.(20)):

$$\lambda_{ii'} = \lambda_0 + \delta\lambda_{ii'}, \qquad |\delta\lambda_{ii'}| \ll \lambda_0. \tag{7}$$

At first we will consider Eqs.(2)–(6) at the symmetric $U_L(n)\times U_R(n)$ interaction (1), i.e. $\delta\lambda_{ii'}=0$. In this case the unknown matrix $\Sigma_{ii'}$ can be taken in a diagonal form: i and i' bases are arbitrary. Solutions we are interested in vary in the number n' of massive states at equal m_1 masses, $n' \leq n$. Flavor indices are preserved in B, all expressions depend only on the chiralities α, β.

Any element ($i = i'$ and $i_1 = i_1'$) of the matrices $(1-A)$ and B can be written as

$$\delta_{\alpha\beta} - A_{\alpha\beta} = A^{(1)}\delta_{\alpha\beta} + A^{(2)}, \qquad B_{\alpha\beta} = B^{(1)}\delta_{\alpha\beta} + B^{(2)}. \tag{8}$$

The equation $(1-A)B = 1$ gives us the part of B participating in Eq.(2)

$$B^{(2)} = -\frac{A^{(2)}}{[A^{(1)} + 2A^{(2)}]A^{(1)}}. \tag{9}$$

The denominators, Eq.(9), represent propagators of both the bound massive scalar and goldstone states (after symmetry breaking). These denominators are equal to

$$\begin{aligned} A^{(1)} + 2A^{(2)} &= \frac{1}{M^2}\left[\Gamma\left(\frac{m_1^2}{M^2}\right)M^2 + \frac{1}{2}\beta(4m_1^2 - q^2)I(m_1^2, q^2)\right], \\ A^{(1)} &= \frac{1}{M^2}\left[\Gamma\left(\frac{m_1^2}{M^2}\right)M^2 + \frac{1}{2}\beta(-q^2)I(m_1^2, q^2)\right], \\ I(m_1^2, q^2) &= \int \frac{d^4p}{\pi^2 i}\frac{1}{(m_1^2 - p^2)[m_1^2 - (p-q)^2]}f, \quad \beta = \frac{\lambda_0 M^2 N_c}{8\pi^2}. \end{aligned} \tag{10}$$

The function $\Gamma(m_1^2/M^2)$ takes the form:

$$\Gamma\left(\frac{m_1^2}{M^2}\right) = 1 - \beta\left(1 - \frac{m_1^2}{M^2}\ln\frac{M^2}{m_1^2}\right) + O\left(\frac{1}{N_c}\right), \tag{11}$$

it represents the expression for "the goldstone mass" (ΓM^2).

The simple loop (Eq.(6)) ($\delta\lambda_{ii} = 0$) contributes the two first terms to Eq.(11). They strictly coincide with the gap equation at $N_c \to \infty$ (the Fig.1a contribution). Preservation of N_c^{-1} terms in the gap equation (2) requires us to take into account N_c^{-1} terms of the same order in Eqs. (10)–(11). They must appear from the terms $N_c^{-1}f_1$, $N_c^{-2}f_2, \ldots$ of Eq.(4). It is impossible to calculate terms with overlapping square divergences in the unrenormalizable model (1), but there is no need to do that. In the denominators (Eq.(10)), the constant $\sim N_c^{-1}$ terms contribute only to the function ΓM^2 representing the goldstone mass. All other possible terms will be smaller: $N_c^{-1}(m_1^2/M^2), N_c^{-1}(q^2/M^2)$. Therefore, after symmetry breaking we should simply take $\Gamma = 0$. Then, the remainder of the Eq.(4) terms $\sim N_c^{-1}, N_c^{-2}, \ldots,$ may be neglected in the gap equation (2). In this way we are consistently

determining the NJL gap equation in the approximation next to the leading N_c one.

Thus, at $\delta\lambda_{ii} = 0$ the second term in rhs (2) is written as:

$$\begin{aligned} \frac{n'}{N_c m_1} F(m_1^2) &= -\frac{2n'}{N_c} \int \frac{d^4q}{\pi^2 i} \frac{m_1^2}{m_1^2 - q^2} \frac{1}{(4m_1^2 - q^2)(-q^2) I(m_1^2, q^2)} \\ &\rightarrow \frac{-2\sqrt{2}\, n'}{\ln(M^2/m_1^2) N_c}. \end{aligned} \tag{12}$$

The integral is calculated at $\ln M^2/m_1^2 \gg 1$. Due to Eq.(9), we have $F < 0$.

In the symmetric case the gap equation is

$$1 - \beta^{-1} = \frac{m_1^2}{M^2} \ln \frac{M^2}{m_1^2} + \frac{n'}{\beta N_c} \left| \frac{F(m_1^2)}{m_1} \right|. \tag{13}$$

The $m_1^2 > 0$ solution exists when

$$\beta > 1. \tag{14}$$

At $N_c \rightarrow \infty$ Eqs.(13),(14) become one loop conditions well known in NJL models ($\beta =$const).

The energy shifts of massive Dirac cellars [1] enable us to compare vacuum energies of different solutions $m_1(n')$. We have

$$E_n^{n'}[m_1(n')] - E_n(0) = -2n' m_1^2(n') 0(\Lambda^2) N_c V, \tag{15}$$

$0(\Lambda^2)$ is the square divergent integral. At $N_c \rightarrow \infty$ a stable solution corresponds to $n' = n$. There exists a region of parameters β and N_c where $n' = 1$ state will have the minimal energy. This region is easy to determine at $M^2/m_1^2 \gg \ln(M^2 m_1^2) \gg 1$. In this case the first term in the rhs (13) may be neglected to obtain the root:

$$\frac{m_1^2(n')}{M^2} \simeq \exp\left[-\frac{2\sqrt{2}\, n'}{(\beta - 1) N_c} \right] \ll 1. \tag{16}$$

The solution of one massive flavor will be stable if $(\beta - 1) N_c \ll 2\sqrt{2}/\ln 2$.

This problem can also be solved by substitution of $\frac{1}{\lambda}\phi^2 - \phi(\bar{\psi}\psi)$ for $\lambda(\bar{\psi}\psi)^2$, eq.(1), via integration over auxiliary scalar fields ϕ [3, 6, 8].

It should be noted that different n' solutions exhibit the following symmetry breakings:

$$SU_L(n) \times SU_R(n) \to SU_L(n-n') \times SU_R(n-n') \times SU_V(n'). \qquad (17)$$

Let us assume $n'=1$ and consider Eq.(2) again. Our aim is to adjust such $\delta\beta_{ii'}$ properties that would necessarily produce the mass hierarchy and mixing in all quark generations: $m_1 \gg m_2 \gg m_3, \ldots$ ($\delta\beta_{ii'} = \delta\lambda_{ii'} M^2 N_c/8\pi^2$).

Well-known (see [9] and refs. therein) is that the hierarchical structure can be explained if lower masses appear as radiative corrections. This mechanism implies that proper mass matrices for up and down quarks $\Sigma^{U,D}_{ii'}$ should be represented as perturbation series in some generation dependent interaction [7]. Therefore, we will also expand the gap equation (2), propagators and B amplitudes ($\delta B = -B_0 \delta A B_0$) with regard of Eq.(7) and the formula:

$$\Sigma_{ii'} = \Sigma_0 + \delta\Sigma_{ii'}, \qquad |\delta\Sigma_{ii'}| \ll \Sigma_0 = \frac{m_1}{n}. \qquad (18)$$

In Eq.(18) we seek $\Sigma_{ii'}$ in the general bases of L, R systems, since such bases appear to be more appropriate in case $\delta\beta_{ii'}$ is unknown. After having eliminated the non-perturbative gap equation (13), we obtain the following form of the $\delta\Sigma_{ii'}$ system:

$$\delta\Sigma_{ii'}\left[1 - \beta^{-1} - \frac{m_1^2}{M^2}\ln\frac{M^2}{m_1^2} + \frac{1}{N_c\beta}f^{(1)}\right] + \frac{\Delta(\delta\Sigma)_{ii'}}{m_1^3}\left[\frac{m_1^2}{M^2}\ln\frac{M^2}{m_1^2} + \frac{1}{N_c\beta}f^{(2)}\right]$$

$$\begin{aligned} &+ \quad \frac{1}{N_c\beta}f^{(3)}\left[\sum_i \frac{\delta\Sigma_{ii'}}{m_1} + \sum_{i'} \frac{\delta\Sigma_{ii'}}{m_1}\right] = -\frac{\delta\beta_{ii'}}{\beta}; \qquad (19) \\ \Delta(\delta\Sigma)_{ii'} \quad &= \quad \delta_{\delta\Sigma}\left\{\Sigma_{ii'} Sp\Sigma^T\Sigma - (\Sigma\Sigma^T\Sigma)_{ii'}\right\}, \end{aligned}$$

where $f^{(i)}$ are the functions of $\ln M^2/m_1^2$. In order to obtain a normal solution, $m_2 \ll m_1$, it is necessary to have a very small asymmetric coupling constant $\delta\beta_{ii'}$:

$$\delta\beta_{ii'} = 0\left(\frac{m_1^2}{M^2}, \frac{1}{N_c}\right)\delta\tilde{\beta}_{ii'}, \qquad |\delta\tilde{\beta}_{ii'}| \leq \sqrt{\frac{m_2}{m_1}} \qquad (20)$$

(if $\delta\beta_{ii'}$ depends on one index, either i or i', $m_2/m_1 \sim |\delta\beta_i|^2$ or $|\delta\beta_{i'}|^2$). The quantity m_2 and mixing properties will be obtained by diagonalization of

Eq.(18), provided $\delta\Sigma_{ii'}$ is a solution of Eq.(19). At $N_c \to \infty$ Eq.(19) has no solutions.

If considered outside the model frame, the restriction (20) may turn out important. The non-perturbative part of B, Eq.(9), preserves flavor, therefore all flavor changings may occur via small non-diagonal constants $\sim N_c^{-1}\delta\beta$ (20). Thus, neutral transitions induced by a new high energy physics could be strongly suppressed.

Due to the factor $0(m_1^2/M^2, N_c^{-1})$ the series for $\beta_{ii'}$ (i.e. for $\lambda_{ii'}$, Eq.(7)) and, hence, the series for $\Sigma_{ii'}$ (which in reality must be induced by β) cannot be usual perturbation expansions. A necessary situation may occur when strong flavor independent forces give rise to a critical phenomenon and only small field components can distinguish flavors [7]. The expansion then takes place in the ratios between small flavor dependent and strong independent fields. In this case the factor $\sim 0(m_1^2/M^2, N_c^{-1})$ can coincide with these ratios. Thus, it becomes apparent that different fields and forces take part in quark spectrum phenomena.

The articles [7] contain the detailed analysis of this quasi-perturbative problem. The resulting point is that practically all qualitative features of the quark mass hierarchy and weak mixings will be reproduced if small flavor dependent fields are of a chiral vector type. The extended publication will present Eq.(19) solution at length.

The author is grateful to Ya.I.Azimov, M.I.Eides, V.Yu.Petrov, and A.V.Yung for clarifying discussions. This work was supported by the Russian Fundamental Investigation Fund (project No.93-02-3145).

References

[1] G.Jona-Lasinio and Y.Nambu, Phys.Rev. **122**, 345 (1961).

[2] Y.Nambu, in: "New Theories in Physics", proceedings of the XI-th Intern.Symp. on Elem.Particle Physics, Kazimirs, Poland, 1988, ed. by Z.Ajduk et al. (World Scientific, Singapore, 1989);
V.A.Miransky, M.Tanabashi, and K.Yamawaki, Phys.Lett. **B221**, 177 (1989).

[3] W.A.Bardeen, C.T.Hill, and M.Lindner, Phys.Rev., **D41**, 1647 (1990).

[4] R.N.Mohapatra and K.S.Babu, Phys.Rev.Lett. **66**, 556 (1991).

[5] M.A.Luty, Phys.Rev. **D48**, 1295 (1993).

[6] R.S.Chivukula, M.Golden, and E.H.Simmons, Phys.Rev.Lett. **70**, 1587 (1993);
W.A.Bardeen, C.T.Hill, and D.U.Jungnickel, Phys.Rev. **D49**, 1437 (1994).

[7] I.T.Dyatlov, Phys.Rev., **D45**, 1636 (1992) (a); **D48**, 5305 (1993) (b).

[8] C.T.Hill, M.A.Luty, and E.A.Paschos, Phys.Rev. **D43**, 3011 (1991).

[9] B.S.Balakrishna, A.L.Kagan, and R.N.Mohapatra, Phys.Lett. **B205**, 345 (1988).

Figure caption

Fig.1. Equation for the fermion self-energy.